Aside – Intermittency

→ Why bother studying percolation?

What is relevance of connections profile?

⇒ Intermittency ?

- turbulence not space filling

- homogeneous but non-uniform excitation

→ Simple Model ?

- β model of K41 cascade

⇒

- fractal dimension

- anomalous exponents

→ Dissipation and Dissipative Structures !?

~ K41 phenomenology suggests uniform distribution of dissipation

↓

ld

i.e. dissipation on scale ld, filling space

~ in reality,

→ distribution of dissipation is variable in intensity

→ not space filling / patchy (intermittent)

i.e.

→ some departure from K41 spectrum concomittant.

⇒ How characterize ?? ⇒ need a phenomenology, first.

⇒ Fractal Intermittency Models
(β - model) ....

Refs: ① Frisch, Sulem, Nelkin
② Frisch, "Turbulence + The Legacy of A.N. Kolmogorov"

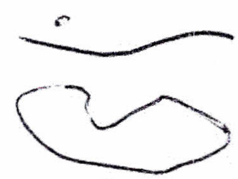
- Fractal? why?
- what does "dimension" mean?

n.b. fractal concepts enable geometric phenomenology

⇒ Dimension

How define dimension?

→ consider structure



[embedded in Cartesian space]

consider

→ covering: N-dimensional cubes (cubes - Cartesian) of size ϵ
(covering set by space structure is embedded in)

→ if $\tilde{N}(\epsilon)$ = # cubes to cover

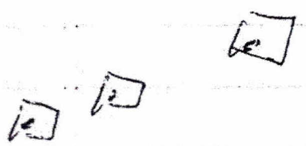
⇒ $$\boxed{D_0 = \lim_{\epsilon \to 0} \frac{\ln \tilde{N}(\epsilon)}{\ln(1/\epsilon)}}$$

Box-Counting Dimension:

$$\boxed{D_0 = \lim_{\epsilon \to 0} \frac{\ln \tilde{N}(\epsilon)}{\ln(1/\epsilon)}} \rightarrow \text{general definition}$$

check:

(a) Finite # points

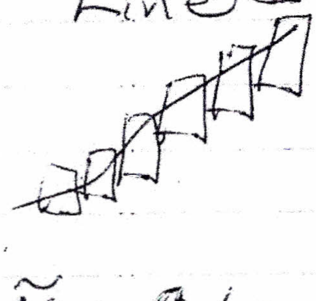


P pts

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln P}{\ln(1/\epsilon)}$$

$$= 0 \quad \checkmark$$

(b) Line — length ℓ



$$D_0 = \lim_{\epsilon \to 0} \frac{\ln \tilde{N}(\epsilon)}{\ln(1/\epsilon)}$$

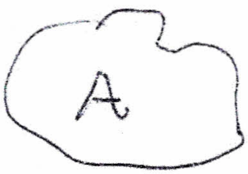
$\tilde{N} = \ell/\epsilon$

$$= \lim_{\epsilon \to 0} \frac{\ln(\ell/\epsilon)}{\ln(1/\epsilon)}$$

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln \ell + \ln(1/\epsilon)}{\ln(1/\epsilon)} \to 1$$

$D_0 = 1$ ✓

c) Closed Curve - Area A



$$D_0 = \lim_{\epsilon \to \infty} \frac{\ln A/\epsilon^2}{\ln(1/\epsilon)}$$

$$= \lim_{\epsilon \to \infty} \frac{\ln A + 2\ln(1/\epsilon)}{\ln(1/\epsilon)}$$

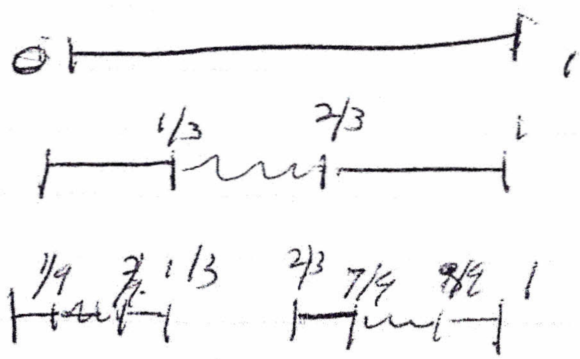
$D = 2$ ✓

Now, something juicier!

~ Middle Third Cantor Set

Define odd set:

Chop out middle 1/3.



for each n, cover with 2^n pieces, $(1/3)^n$ length:

$\tilde{N}$ above 2^n; ϵ below $(1/3)^n$

$$D_0 = \lim_{\substack{n\to\infty \\ (\epsilon\to 0)}} \frac{\ln 2^n}{\ln\left(1/\left(\frac{1}{3}\right)^n\right)}$$

$$= \lim_{n\to\infty} \frac{n \ln 2}{n \ln 3}$$

$$\boxed{\begin{aligned} D_0 &= \ln 2/\ln 3 \\ D_0 &= .63\ldots \end{aligned}}$$

- fractal dimension
- $0 < D_0 < 1$
- embedded in $D = 1$ space $D_0 < D_{embed}$
- note $\leftrightarrow$ $D \leftrightarrow$ power law

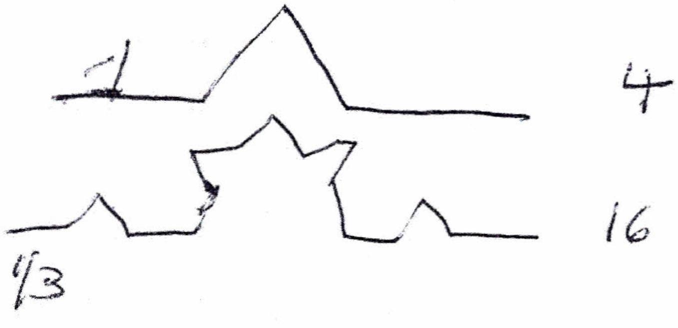
i.e. $\tilde{N}(\epsilon) \sim \epsilon^{-D_0}$

box counting dimension

$\boxed{\text{Fractals are self-similar.}}$

 [redacted]

~ Could also encounter Koch Curve



i.e. every

upon iteration

$$D_0 = \lim_{\epsilon \to 0} \frac{N(\epsilon)}{\ln(1/\epsilon)}$$

$$= \lim_{\substack{\epsilon \to 0 \\ n \to \infty}} \frac{4^n}{\ln\left[1/(1/3)^n\right]}$$

$$= \lim_{n \to \infty} \frac{2n \ln 2}{n \ln 3} = \frac{2 \ln 2}{\ln 3}$$

$$\boxed{\begin{array}{l} D_0 = 2\ln 2/\ln 3 \\ D_0 \sim 1.2\ldots \end{array}}$$

- here example of a fractal which thickens, i.e. $1 < D < 2$. Not embedded in 2D.

- akin "coast-of-Britain" problem (Richardson '61, Mandelbrot ...)

i.e. increased resolution reveals
longer, more convoluted coastline.

$$\tilde{N}(\epsilon) \sim \epsilon^{-D_0}$$

⇒ rougher on smaller scale...
(N increases with $\epsilon \downarrow$).

Why Fractals?

→ self-similar structures with
dimension < dimension of
embedding space (i.e. 3)

⇒ natural candidates to describe:

- intermittent dissipation events
- geometry of dissipative
structures in intermittent
turbulence / cascades

→ cascade is due hierarchical, embedded
process; dissipative structure
does not fill space

D_0 ⇒
- intermittency correction to K41
spectrum!

→ The idea:

- fractal structure is picture/phenomenology of observed departure from K41 spectrum.

- trends of scalings → plausible (i.e. fit)

but

- theory, based on NS Eqn, does not "predict" D_0!

N.B. { Geometrically/symmetry motivated phenomenology is extremely useful i.e. Landau–Ginzburg, etc.

which brings us to:

→ **β - model** (Frisch, Sulem, Nelkin)

→ basic idea: (Mandelbrot)

- cascade active region is self-similar fractal structure, with $D < 3$.
- dissipation events are 'patchy'
- forces correction to K41.

→ Analysis

- why intermittency? ⇔ physics of cascade.

⇒ vortex stretching is very nonlinear

$$\partial_t \underline{V} + \underline{V} \cdot \nabla \underline{V} = - \nabla \overset{\text{enthalpy}}{w} + \nu \nabla^2 \underline{V}$$

$$\partial_t \underline{V} = - \nabla \left(w + \frac{\underline{V}^2}{2} \right) + \underline{V} \times \underline{\omega} + \nu \nabla^2 \underline{V}$$

$\underline{\omega} = \nabla \times \underline{V}$ → vorticity (key physics)

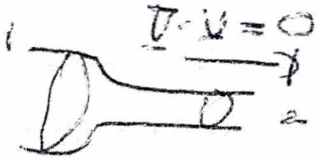
$\Rightarrow \quad \underline{\nabla} \times \quad \Rightarrow$

$$\boxed{\partial_t \underline{\omega} = \underline{\nabla} \times (\underline{v} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}}$$

vorticity advection eqn.

$$\frac{d\underline{\omega}}{dt} = \underline{\omega} \cdot \underline{\nabla}\, \underline{v} + \nu \nabla^2 \underline{\omega}$$

↳ vortex stretching



$$\sim \quad \omega^2 + \nu \nabla^2 \underline{\omega}$$

↑
heuristic only

Kelvin Thm
$\oint \underline{v} \cdot d\ell = \text{const.}$

$\Rightarrow$ fast (nearly explosive) growth of vorticity, enstrophy $\langle \omega^2 \rangle$ produced, to dissp.n.

$\Rightarrow$ bursts, etc.

$\Rightarrow$ vortex stretching feeds on self $\Leftrightarrow$ localized process.

$\rightarrow$ so, "patchy" "embedded" cascade:
↳ occupation factor

$$\bar{\epsilon} \sim \beta_n \frac{V_n^3}{\ell_n}$$

↑
mean dissp.n. rate

$\beta_n \equiv$ { fraction of space active on nth step cascade

N.B.

- $\oint \underline{v}\cdot d\underline{l} = \int \underline{\omega}\cdot d\underline{a} = \text{const}$

$\omega_1 r_1^2 \sim \omega_2 r_2^2 \Rightarrow$ vorticity increases on small scale

- N.B. analogy

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \eta \underline{J}$$

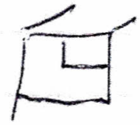
$$\nabla\times\underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t}$$

$$\Rightarrow \partial_t \underline{B} = \nabla\times\left(\frac{\underline{v}\times\underline{B}}{c}\right) + \eta\nabla^2\underline{B}$$

β_n ? → fraction of volume active in n^{th} step of cascade

N.B. – if each eddy scale $\ell \to \ell/2$ per step

then # children to fill space per step

is $2^3 = 8$ i.e. 

$$\Rightarrow \quad \beta = N/2^3 < 1$$

↑ # off-spring

occupation reduction factor.

$$\beta_n = (\beta)^n = (N/2^3)^n \quad \to n \text{ steps.}$$

→ now, interpretation only

$$N \equiv 2^D \qquad D < 3$$

(simply an interpretation)

↑ box counting dimension

$$\boxed{\beta_n = \left(2^{D-3}\right)^n}$$

So, taking mean energy balance

$$\bar{\epsilon} = \beta_n \frac{v_n^3}{l_n} \qquad\qquad \beta_n = 2^{n(D-3)} = \left(\frac{l_0}{l_n}\right)^{D-3}$$

$$= \left(\frac{l_n}{l_0}\right)^{3-D} \frac{v_n^3}{l_n}$$

$\Rightarrow$

$$v(l_n) \sim (\bar{\epsilon} l_n)^{1/3} \left(l_n/l_0\right)^{-1/3(3-D)}$$

correction due $D \neq 0$

$\Rightarrow$ induces explicit l_0.

Fraction of active

$$E_n \sim \beta_n v(l_n)^2 \sim \underbrace{\bar{\epsilon}^{2/3} l_n^{2/3} \left(l_n/l_0\right)^{-2/3(3-D)}} \underbrace{\left(\frac{l_n}{l_0}\right)^{(3-D)}}$$

velocity in active region

$$\sim \bar{\epsilon}^{2/3} l_n^{2/3} \left(l_n/l_0\right)^{(3-D)/3}$$

eqn 80

$$\boxed{\begin{aligned} E(\ell_n) &\sim \bar{\epsilon}^{2/3} \ell_n^{2/3} (\ell_n/\ell_0)^{(3-D)/3} \\ E(k) &\sim \bar{\epsilon}^{2/3} k^{-5/3} (k \ell_0)^{-\frac{1}{3}(3-D)} \end{aligned}}$$

→ correction to K41, in proportion $3-D$

→ slight <u>steepening</u> of spectrum

can deduce effective dimension from fit to spectral data.

Finally, dissipation scale changes;

<u>i.e.</u> $\dfrac{\nu}{\ell_d^2} = \dfrac{V(\ell_d)}{\ell_d}$ $\qquad R_e \sim \dfrac{\ell_0 V_0}{\nu}$

<u>but</u> $\qquad \bar{\epsilon} \sim V_0^3/\ell_0$

$$V(\ell_d) \sim \bar{\epsilon}^{1/3} \ell_d^{1/3} (\ell_d/\ell_0)^{-\frac{(3-D)}{3}}$$

$$\Rightarrow \boxed{l_d \sim l_0 (Re)^{-3/1+D}}$$

$$Re = \frac{l_0 V_0}{\nu} = \frac{\epsilon^{1/3} l_0^{4/3}}{\nu}$$

$D = 3$

$$l_d \sim l_0 \left(\frac{\epsilon^{1/3} l_0^{4/3}}{\nu}\right)^{-3/4}$$

$$\sim \cancel{\frac{l_0}{l_0}} \, \epsilon^{-1/4} \, \nu^{+3/4}$$

$$l_d \sim \nu^{3/4} / \epsilon^{1/4} \, , \quad D = 3.$$

modified for $D < 3$.

More Intermittency

A.)

→ Why the Fractology and what do we get from β - Model ?

- $\boxed{\text{higher moments are a more severe probe of small scale structure of turbulence than energy is!}}$

Recall K41 ⇒ $\delta v(\ell) \sim \epsilon^{1/3} \ell^{1/3}$

$$\therefore \langle \delta v(\ell)^p \rangle \sim \epsilon^{p/3} \ell^{p/3}$$

so normalizing:

$$\langle \delta v(\ell)^p \rangle \Big/ \langle \delta v(\ell)^2 \rangle^{p/2} \sim 1$$

normalized moments all independent of scale! → Testable Prediction

So, what of higher moments, i.e. $p > 2$?

Special interest in:

$\rightarrow$ $p = 3$ – skewness S (measure of symmetry)

why? turbulence $\rightarrow$ statistical approach/picture

naively: Gaussian distribution (i.e. random)

so $S \rightarrow 0$.

but:

$$\partial_t E \sim \partial_t V^2 \sim V^3 \quad ; \quad \underline{\text{so}}$$

net energy transfer in cascade, and

$$\langle V^3 \rangle \neq 0.$$

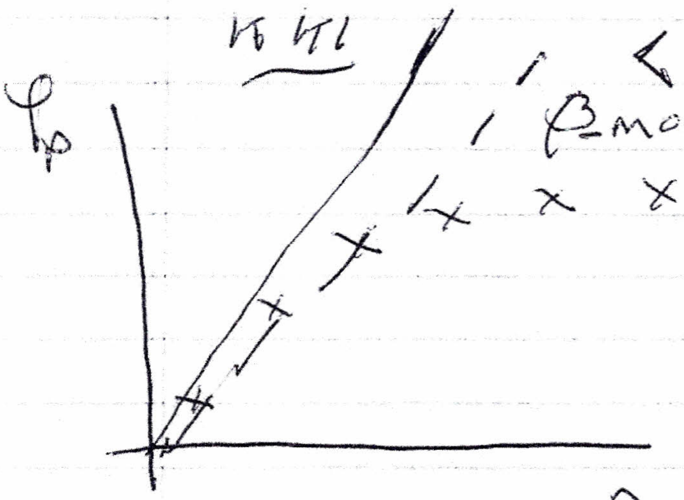
Similarly, $p = 4$ – kurtosis K

$K = \langle \delta V^4 \rangle / \langle \delta V^2 \rangle^2 \rightarrow 3$ for Gaussian process. $\Rightarrow$ measure of importance/weight of <u>tails</u> of distribution

$\Rightarrow$

$K \gg 1$ is indicative of strong correlations and non-Gaussian behavior, and fat tails.

→ The Data (mostly in $\underline{\text{time}}$, i.e. $\langle V_x(t)^p \rangle$ etc.)



⇒ reality departs K41!

→ What does β-model predict?

$$\langle |\delta v(\ell_n)|^p \rangle \sim \overset{\text{volume factor}}{\beta_n}\, \underbrace{[\delta v(\ell_n)]^p}_{\text{set by cascade scaling}}$$

$$\sim \bar{\epsilon}^{\,p/3}\, \ell_n^{\,p/3} \underbrace{(\ell_n/\ell_0)^{\zeta_p}}_{\text{exponent of intermittency correction}}$$

$$\zeta_p = \frac{1}{3}\overset{\text{wt factor}}{(3-D)}\,\underbrace{(3-p)}_{\delta v^p}$$

So, normalization $\Rightarrow$

$$a_p(\ell_n) \sim \langle \delta v(\ell_n)^p \rangle / \langle \delta v(\ell_n)^2 \rangle^{p/2}$$

plugging in $\Rightarrow$

$$\boxed{\begin{aligned} a_p(\ell_n) &\sim (\ell_n/\ell_0)^{\varepsilon_p} \\ \varepsilon_p &= 1/2\,(3-D)(2-p) \end{aligned}}$$

n.b.
$p > 2$
$\varepsilon_p < 0$

In particular:

$$S \sim \langle \psi^3 \rangle / \langle \psi^2 \rangle^{3/2}$$

$\psi \sim \nabla v$

$$\sim Re^{3(3-D)/2(1+D)}$$

taking $\ell_n \sim \ell_d$ as effect maximal.

$$K \sim S^2$$

Note:

- departure from K41 strongest at smaller scales
  $\Rightarrow$ 'focus' of cascade strongest

- β model $\Rightarrow$ stages in cascade have "memory" of initial scale l_0

  $\Rightarrow$ l_0 explicit, beyond ϵ.

- $D = 2.8$ is reasonable data fit

  $\Rightarrow$ dissipative structure is highly convoluted sheets.

- ζ_p departs β-model as $p \uparrow$

  $\Rightarrow$ Multi-fractal model; i.e.

  β-Model $\rightarrow$ single dissipative structure of dimension D.

  Multi-fractal $\rightarrow$ multiple dissipative structures, different

$\Rightarrow$ connection to Navier-Stokes equation and dynamics is increasingly obscure ....

[redacted]

— a natural question:

→ have argued that intermittency

⟺ departure from simple, self-similarity scaling

⇒ manifested as l_i/l_0 "memory" in structure function

→ have also stressed analogy between

self-similarity in space (Blast wave)

vs self-similarity in scale (K41).

∴

→ so, what is analogue of intermittency for space-time similarity, i.e.

K41 ⟷ β- model

as

Spatio-temporal self-similarity ⟷ ?

$\Rightarrow$ Memory of initial condition!

i.e. $f \rightarrow f(\underbrace{r/\rho(t)}_{\text{self-sim. variable}}, r_0)$

Note for Sedov-Taylor effectively ignored initial radius of blast!

$\Rightarrow$ $\left\{ \begin{array}{l} \text{\underline{See Barenblatt}, "Scaling"} \\ \text{Chapt. 3} \end{array} \right.$

Now, one can go further, and calculate:

$\langle \tilde{\epsilon}^2 \rangle \rightarrow$ mean square fluctuating dissipation

but $\epsilon \sim \nu \langle (\nabla \tilde{v})^2 \rangle$

so $\langle \tilde{\epsilon}^2 \rangle \sim \nu^2 \langle (\nabla \tilde{v})^2 (\nabla \tilde{v})^2 \rangle \sim \nu^2 \beta_{\ell_d} \dfrac{\tilde{v}(\ell_d)^4}{\ell_d^4}$

if normalize:

$$\left. \begin{array}{l} \langle \tilde{\epsilon}^2 \rangle / \langle \epsilon \rangle^2 \sim \cancel{\nu^2} K \\ \sim \cancel{\nu^2} Re^{3(3-D)/(1+D)} \end{array} \right\} \text{Kurtosis!}$$

Can also address:

$$\boxed{\langle \epsilon(\underline{r})\,\epsilon(\underline{r}+\underline{\ell})\rangle \rightarrow \text{dissipation correlation}}$$

Now,

$$\langle \epsilon(\underline{r})\,\epsilon(\underline{r}+\underline{\ell})\rangle \sim \langle \epsilon\rangle^2\, \text{Prob.}(\underline{r}, \underline{r+\ell} \text{ belong to } n\text{-eddy})$$

$$\langle \epsilon \rangle \sim v_n^3/\ell_n$$

$$\sim v(\ell_n)^3/\ell_n$$

⇒ allowing for:
- packing
- if correlated by ℓ_n then correlated by all larger eddies

⇒

$$\langle \epsilon(r)\,\epsilon(r+\ell)\rangle \sim \sum_{n=0}^{\ell_n \lesssim \ell} \left(\frac{v_n(\ell_n)}{\ell_n}\right)^2 \beta_n$$

$$\sim \bar{\epsilon}^2 (\ell/\ell_0)^{-(3-D)}$$

In particular,

$$\boxed{\langle \epsilon(r)\, \epsilon(r+l_d) \rangle \sim \bar{\epsilon}^2 \left(l_d / l_0 \right)^{(D-3)}}$$

$\Rightarrow$ strong correlation in dissipation at dissp'n. scale.